Using presentations to describe groups

Gilbert Baumslag

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1 Presenting semi-direct products

Suppose that G is a group and that U and T are subgroups of G. In addition, suppose that $U \trianglelefteq G$, that $G = TU$ and that $T \cap U = 1$. Notice that T can be equipped with a homomorphism ϕ into the automorphism group of U where

$$\phi : t \mapsto \hat{t},$$

where

$$\hat{t} : u \mapsto t^{-1}ut \ (t \in T, u \in U).$$

Then it follows that

$$G \cong U \rtimes_\phi T. \qquad (*)$$

Exercise 1.1 *Convince yourselves that (*) makes sense.*

Exercise 1.2 *Suppose now that G is as above, with its subgroups U and T. Furthermore, suppose that U and T are presented as follows:*

$$U =< X; R >, \ T =< Y; S > .$$

Now each of these presentations come equipped with presentation maps and if we identify the elements of X and Y with their respective images in U and T, then

$$y^{-1}xy = w_y(\underset{\rightarrow}{x}),$$

where here $w_y(\underset{\rightarrow}{x})$ denotes a word in the generators X of U which is equal in U to the element in U obtained by conjugating x by y in G. Verify that

$$G =< X \cup Y; R \cup S \cup \{y^{-1}xy = w_y(\underset{\rightarrow}{x}) \mid y \in Y, x \in X\}.$$

The point here is that the constructed semi-direct product provides a model of the group presented in this way.

Exercise 1.3 *Find a presentation of* $A \wr T$.

2 Free products

Suppose that

$$A = < X; R >, \ B = < Y; S > .$$

Then

$$G = < X \cup Y; R \cup S >$$

is termed the free product of A and B and is denoted by writing

$$G = A * B.$$

Exercise 2.1 1. *Prove that* A *and* B *are embedded in* G, *i.e., the obvious mappings of* A *and* B *into* G *are monomorphisms.*

2. *By 1 we can identify* A *and* B *with their images in* G. *Observe then that*

$$G = gp(A, B).$$

3. *Prove that*

$$A \cap B = 1.$$

4. *Define a product*

$$p = s_1 \dots s_n \ (n \geq 1, s_j \in A \cup B)$$

to be strictly alternating if no $s_j = 1$ *and if successive* s_j *come from different* factors, *i.e., if* $s_j \in A$ *then* $s_{j+1} \in B$ *and vice-verse. Prove that every strictly alternating product is not equal to 1. You will certainly need to consult a text-book to verify this - have a look at Magnus, Karrass and Solitar.*

5. *Prove that the free product of two groups of order 2 is the infinite dihedral group.*

6. *Suppose that* $A = < a; a^2 = 1 >$ *and* $B = < b; b^3 = 1 >$. *Prove that*

$$H = gp(ab, ab^2)$$

is a free group freely generated by ab *and* ab^2.

7. *Suppose that* $G = A * B$. *Prove that*
$$H = gp([a, b] \mid a \in A, b \in B)$$
is free on the elements $[a, b]$ *where now* a *and* b *are non-trivial.*

3 Free products with amalgamation

Free products with amalgamation are generalizations of free products. The data here is similar to that in the case of free products. To this end suppose that $A =< X; R >$, that $B =< Y; S >$ and that H and K are subgroups of A and B and that $\phi : H \longrightarrow K$ is an isomorphism. Furthermore suppose that H is generated by the elements $h_i(\underset{\longrightarrow}{x}) \mid i \in I$, where each h_i is expressed as an X-word and that K is generated by the elments $k_i(\underset{\longrightarrow}{y}) \mid i \in I$, where each k_i is expressed as a Y-word and such that $h_i\phi = k_i$. Then the group G given by the presentation
$$< X \cup Y; R \cup S \cup \{h_i(\underset{\longrightarrow}{x})(k_i(\underset{\longrightarrow}{y}))^{-1} \mid i \in I\} >$$
is termed the generalized free product of A and B with H amalgamated with K according to the isomorphism ϕ or simply an amalgamated product of A and B with H and K amalgamated. We use the following notation for such an amalgamated product:
$$G = \{A *_{H=K} B\} \text{ or } G = A *_H B.$$

1. Prove that A and B are embedded in $\{A *_{H=K} B\}$. So we can identify A and B with their images in G. This is non-trivial and you will have to consult Magnus, Karrass and Solitar for a proof.

2. In $\{A *_{H=K} B\}$ we find that if $h \in H$, then $h = h\phi$ and hence that $A \cap B = H$, which explains the notation $G = A *_H B$.

3. A product
$$p = s_1 \dots s_n \ (n \geq, s_j \in A \cup B)$$
is said to be strictly alternating if no $s_j \epsilon H$ (or therefore K) and if successive s_j come from different *factors*, i.e., if $s_j \in A$ then $s_{j+1} \in B$ and vice-verse. Prove that every strictly alternating product is not equal to 1. Again you may well have to consult Magnus, Karrass and Solitar for a proof.

4. Prove that if
$$s_1 \dots s_n = t_1 \dots t_m,$$
both of these products are strictly alternating, then s_1 and t_1 belong to the same factor, $s_1 H = t_1 H$ and $n = m$.

5. If we choose S to be a left transversal (i.e., a complete set of representatives of the left cosets aH of H in A containing 1, and T to be a left transversal of H in B, prove that every element $g \in G,\ g \notin H$ can be uniquely written as a stricly alternating product

$$g = u_1 \ldots u_n h,$$

where the $u_i \in S \cup T$ and $h \in H$.

6. If $A_1 \leq A$ and $B_1 \leq B$ and $A_1 \cap H = B_1 \cap H$, prove that

$$gp(A_1, B_1) = \{A_1 * B_1; A_1 \cap H\}.$$