Basic groups, their presentations and group-theoretic constructions

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1 Some basic groups and their presentations

1. *The cyclic groups of order n.* These are the groups C_n with presentations

$$C_n =< a; a^n > .$$

C_∞ is the infinite cyclic group.

2. *Direct products.* As usual if A and B are a pair of groups then the direct product

$$D = A \times B = \{(a, b) \mid a \in A, b \in B\}$$

of the groups A and B is set-theoretically the cartesian product of A and B with coordinate-wise multiplication.

There is an obvious way of presenting such a product. If

$$A =< X; R >, \ B =< Y; S >$$

then

$$D = A \times B =< X \cup Y; R \cup S \cup \{[x, y] \mid x \in X, y \in Y\}\} > .$$

If we identify $a \in A$ with $(a, 1)$ and $b \in B$ with $(1, b)$, then A and B become normal, commuting subgroups of D and

1. $D = gp(A, B)$;

2. $A \cap B = 1$;

3. $[A, B] = 1$.

It follows that every element of D can be written uniquely in the form ab, where $a \in A, b \in B$ and

$$ab.a'b' = aa'.bb'.$$

There is a similar construction with an arbitrary number of factors. To this end, suppose that I is a non-empty index set and that $\{A_i \mid i \in I\}$ is a family of groups A_i indexed by the elements $i \in I$. Then we can construct two new, if I is infinite, groups from the A_i as follows. The first is called the *restricted direct product* while the second is called the *unrestricted direct product*.

Definition 1.1 *Let $\{A_i \mid i \in I\}$ be a family of groups indexed by the elements $i \in I$. Let*

$$\prod_{i \in I} A_i = \{f : I \longrightarrow \{\bigcup_{i \in I} A_i \mid f(i) \in A_i, f(i) \neq 1 \text{ for finitely many i}\}\}.$$

Then if $f, g \in \prod_{i \in I} A_i$ we define fg by

$$(fg)(i) = f(i)g(i) \ (i \in I).$$

We call a $\prod_{i \in I} A_i$ the restricted direct product of the A_i. If we remove the restriction that the functions f involved take on only finitely many non-trivial values, the resultant group is termed the unrestricted direct product and is denoted by $\overline{\prod}_{i \in I} A_i$.

Exercise 1.2 *Find a presentation for $\prod_{i \in I} A_i$ given presentations for the A_i.*

Definition 1.3 *We call a group free abelian if it is (isomorphic to a) restricted direct product of infinite cyclic groups.*

Exercise 1.4 *Prove that the cardinality of the set of factors that make up a free abelian group is an invariant of that group.*

Exercise 1.5 *The rigid motions of a regular n-gon forms a group under composition called the dihedral group. Find presentations for the dihedral groups.*

Exercise 1.6 *If D is the direct product of the two groups A and B, can you prove that D is finitely presented if and only both A and B are?*

Exercise 1.7 *Given a decomposition of a finitely generated abelian group A into a direct product of cyclic groups write down a presentation for A.*

Exercise 1.8 *Prove that a finite group is finitely presentable.*

Exercise 1.9 *Find all of the groups of order 4 and order 8.*

2 Automorphism groups

Definition 2.1 *A homomorphism α of a group G into itself is called an endomorphism. If α is one-to-one and onto, it is called an automorphism. The set of automorphisms of a group G forms a group under composition called the autmorphism group of G and denoted by Aut(G).*

Given a group G it is not always easy to compute its automorphism group of G. Sometimes one can check directly that a map from G into itself is an automorphism. In the event that G is given by generators and defining relations, von Dyck's lemma enables one to check that a map from the generators of G into G defines a homomorphism of G into G and thence whether it is a surjective endomorphism. In order to verify that it is also one-to-one, this is often done by inspection or else by trying to find whether this surjection has an inverse.

Examples 2.2

1. *$Aut(C_n)$ is an abelian group of order $\phi(n)$, where ϕ is the Euler totient function.*
2. *If A is an abelian group, then the mapping*
$$\phi : a \mapsto a^{-1} \ (a \in A)$$
is an automorphism.
3. *The automorphism group of $C_2 \times C_2$ is the symmetric group of degree 3.*
4. *The automorphism group of a free abelian group of rank n is (isomorphic to) the group of $n \times n$ integral matrices of determinant ± 1.*
5. *Find the automorphism groups of the the two non-abelian groups of order 8.*
6. *Have a look at MKS for a discussion of the automorphism group of a free group of rank n.*

2.1 Semi-direct products

Definition 2.3 *Let U and T be a pair of groups and let $\phi : T \longrightarrow Aut(U)$. We now form the cartesian product $E = T \times U$ and define a binary operation on E as follows:*
$$(t, u)(t', u') = (tt', uu(t'\phi)u') \ (u, u' \in U, t, t' \in T).$$
E is called the semi-direct product of U by T and is denoted as follows:
$$E = U \rtimes_\phi T \text{ or } U \rtimes T.$$

The mappings

$$u \mapsto (1,u), t \mapsto (v,1) \ (u \in U, t \in T)$$

define monomorphisms from U and T into E and if we identify U and T with their images in E, then we find that

$$E = TU, \ U \trianglelefteq E, \ U \cap T = 1.$$

It follows that every element $e \in E$ can be written uniquely as a product $e = tu$ $(t \in T, u \in U)$ and that if $s \in T, v \in U)$, then

$$tu.sv = (ts)(u(s\phi)).$$

Hence

$$u^t = u(t\phi).$$

What this means is that conjugation of the elements u in U by the elements t in T in E are the result of acting on U by the automorphisms $t\phi$ that ϕ sends the elements of T to.

Exercises 2.4 *1. Let* $U =< a, b; a^2 = b^2 = (ab)^2 = 1 >$, $T =< t : t^2 = 1 >$. *Let* $\alpha \in (AutU)$ *be defined by*

$$\alpha : a \mapsto b, \ b \mapsto b.$$

Define $\phi : T \longrightarrow Aut(U)$ *by* $t\phi = \alpha$. *Form* $E = U \rtimes_\phi T$. *Prove that* E *is the dihedral group of order* 8.

2. *Let* U *be as above, let* T *be the infinite cyclic group on* t, *let* α *be as above and let* ϕ *again be as above. Compute the center* C *of* $E = U \rtimes_\phi T$ *and prove that* E/C *is the dihedral group of order 8.*

3. *Find an automorphism of order 3 of* $U =< a, b; a^2 = b^2 = (ab)^2 = 1 >$ *and use this to make a group of order 12.*

4. *Let* U *be free abelian of rank 2 and let* $T = GL(2, \mathbb{Z})$. *Form the semi-direct product of* U *by* T *noting that* $GL(2, \mathbb{Z})$ *can be identified with* $Aut(U)$.

5. *Make a non-abelian group of order* p^3, *using semi-direct products.*

6. *Find a non-abelian semi-direct product of two infinite cyclic groups.*

7. *Let*

$$U =< a_0, a_1, \ldots, a_n, \ldots; a_1^2 = a_0, a_2^2 = a_1, \ldots, a_n^2 = a_{n-1}, \cdots > .$$

Then the mapping

$$\alpha : u \mapsto u^2 \ (u \in U)$$

is an automorphism of U. *Let* T *be infinite cyclic on* t *and let* $\phi : T \longrightarrow Aut(U)$ *be the homomorphism which sends* t *to* α. *Form* $E = U \rtimes_\phi T$. *Prove that* E *is a 2-generator group.*

8. *Let* A *and* T *be an arbitrary pair of groups and let*

$$U = \{f : T \longrightarrow A \mid f(t) = 1 \text{ for all but finitely many } t \in T\}.$$

Then U *is the restricted direct product of a family of groups isomorphic to* A *and indexed by the elements of* T. *For each element* $s \in T$ *define*

$$\alpha_s : U \longrightarrow U$$

defined by

$$f\alpha_s(t) = f(st^{-1}) \ (t \in T).$$

Then $\phi : t \mapsto \alpha_t$ *is a homomorphism of* T *into the automorphism group of* U. *The semi-direct product* $E = U \rtimes_\phi T$ *is called the wreath product of* A *by* T *and is denoted* $A \wr T$. *Prove that the center of* $A \wr T$ *is trivial if* $A \neq 1$ *and* T *is infinite.*